SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

November 2002

First HSC Assessment Task

Mathematics

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used
- All necessary working should be shown in every question

Total Marks – 60

- All Questions may be attempted
- Each Question is worth 12 marks

Examiner – *A.M.Gainford*

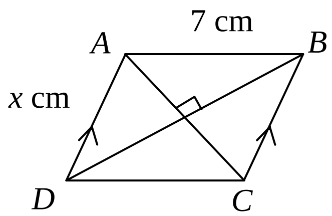
Question 1. (12 Marks) (**Start a new booklet**.)

(a) Calculate $(5\cdot 413698\times 10^{12})\div(2\cdot 910064\times 10^{17})$, giving your answer in scientific notation, correct to 6 significant figures. **1**

(b) Calculate the probability of obtaining a total of 8 when two standard dice are rolled. **1**

(c) Factorise completely $2x^2+2x-12$. **1**

(d) In the figure *ABCD*, $AB \parallel CD$, $AD \parallel CB$ and $AC \perp BD$. If $AB = 7\text{cm}$, $AD = x\text{cm}$, find the value of *x*. **1**



(e) Write an equation for the parabola with vertex (0, 0) and focus (0,2). **1**

(f) Sketch on the number plane the graph of the function $y = \log_3 x$ in the domain $0 < x \le 9$. **1**

(g) Find the 24th term of the arithmetic series $7 + 4\frac{1}{2} + 2 + \ldots$ **1**

(h) Given that $\log_a 10 = 2\cdot 094$ and $\log_a 2 = 0\cdot 6309$, find $\log_a 500$. **1**

(i) Find $\lim\limits_{x\to 3}\dfrac{x^2-9}{x-3}$. **2**

(j) Find the values of *k* for which x^2+kx+2 is positive definite. **2**

Question 2. (12 Marks) (**Start a new booklet**.)

(a) Find and simplify the derivative of each of the following: **6**

(i) $x^3 - 3x^2 + x - 2$

(ii) $\sqrt{1-x}$

(iii) $x\sqrt{1-x^2}$

(iv) $\dfrac{x+2}{1-x}$

(b) Consider the parabola with equation $y = \frac{1}{4}x^2 - x$ **4**

(i) Write the equation in the form $(x-h)^2 = 4a(y-k)$.

(ii) State the vertex and focus.

(iii) Sketch the curve.

(c) (i) Evaluate $\sum_{r=1}^{5} 2r - 1$ **2**

(ii) Express the geometric series $1 + 2 + 4 + 8 + \ldots + 256$ in sigma notation.

Question 3. (12 Marks) (**Start a new booklet**.)

(a) Seventy-five tagged fish are released into a dam known to contain fish. Later a sample of forty-two fish was netted from this dam and then released. Of these forty-two fish it was noted that five were tagged. **1**

Estimate the number of fish in the dam.

(b) The vertices of a triangle are $A(3,4)$, $B(-2,2)$ and $C(5,-3)$. **8**

(i) Find the coordinates of *D*, the midpoint of the side *BC*.

(ii) Write down the equation of the side *AB*.

(iii) Find the equation of the line through *C* parallel to *AD*.

(iv) Find the coordinates of *E*, the point of intersection of the two lines descrbed in parts (ii) and (iii).

(c) (i) Find the value of *m* for which $\log(9^m) = \log 3 - \log\sqrt{3}$. **3**

(ii) Evaluate $\log_b a \times \log_a b$.

Question 4. (12 Marks) (**Start a new booklet**.)

(a) Given the expression $2x^2 + 4x - 1$: **2**

(i) Find the value of x when the expression has its minimum value.

(ii) State the minimum value of this expression

(b) Find the co-ordinates of the point on the curve $y = x^3 + 3x^2 + 3x - 7$ where the gradient of the tangent is zero. **2**

(c) The twelfth term of an arithmetic series is 2, and the fifteenth term is -4. Find the first term and the common difference. **2**

(d) (i) Write a quadratic equation with roots -5 and 7. **3**

(ii) Solve the quadratic inequality $x^2 - 2x - 3 \geq 0$.

(e) Consider the recurring decimal fraction $F = 0 \cdot 4\dot{3}\dot{7}$. **3**

(i) Express F as an infinite sum of terms, all but the first of which form a geometric series.

(ii) Hence or otherwise express F as a common fraction in lowest terms.

Question 5. (12 Marks) (**Start a new booklet**.)

(a) Solve the equation $x^4 - 3x^2 + 2 = 0$. **2**

(b) Consider the curve with equation $y = x - \frac{1}{x}, \quad x > 0$: **3**

(i) Find the gradient of the tangent at the point on the curve where $x = 2$.

(ii) Write in general form the equations of the tangent and normal to the curve at this point.

(c) An amount \$$A$ is borrowed at r% per annum reducible interest, calculated monthly. The loan is to be repaid in equal monthly instalments of \$$M$. **7**

Let $R = \left(1 + \frac{r}{1200}\right)$ and let \$$B_n$ be the amount owing after n monthly repayments have been made.

(i) Show that $B_n = AR^n - M\left(\frac{R^n - 1}{R - 1}\right)$.

Pat borrows \$90 000 at 8% per annum reducible interest, calculated monthly. The loan is to be repaid in 96 equal monthly instalments.

(ii) Show that the monthly repayments should be \$1272·30.

(iii) With the twenty-fourth payment, Pat pays an additional \$10000, so this payment is \$11272·30. After this, repayments continue at \$1272·30 per month. How many more repayments will be needed?

This is the end of the paper.

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Sample Solutions

Question 1

1 a) $5.413698 \times 10^{12} \div 2.910064 \times 10^{17}$

(6 s.f) 1.86037×10^{-5} (1)

b) $5/36$ (1)

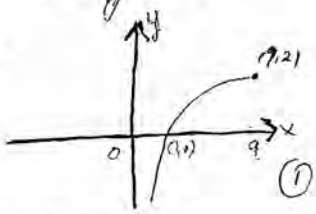
c) $2(x^2 + x - 6)$ (1) : factorise completely

$2(x+3)(x-2)$

d) $x = 7$ (1)

e) $x^2 = 8y$ (1)

f)



(1)

g) $d = -2\frac{1}{2}$, $a = 7$

$T_{24} = a + 23d$

$= 7 + 23 \times -\frac{5}{2}$

$= 7 - 57\frac{1}{2}$ (1)

$= 50\frac{1}{2}$

h) $\log 500 = \log \frac{1000}{2}$

$= \log 1000 - \log 2.$

$= 3\log 10 - \log 2$

$= 6.282 - 0.6309$

$= 5.651$ (1)

i) $\lim \frac{(x+3)(x-3)}{x-3} = \lim x+3 = 6$ (2)

j) $k^2 - 8 < 0$

$-\sqrt{8} < k < \sqrt{8}.$

Q1.

a) 1.86034×10^{-5}

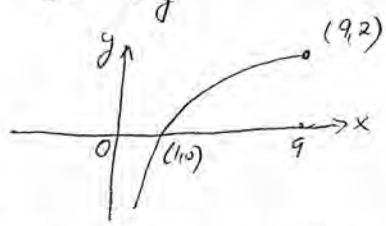
b) $5/36$

c) $2(x+3)(x-2)$

d) $x = 7$

e) $x^2 = 8y$

f)



g) $T_{24} = a + 23d$ $\quad -50\frac{1}{2}$

h) 5.6511

i) 6 (2)

j) $-\sqrt{8} < k < \sqrt{8}$. (2)

Question 2

(a) (i) $\frac{d}{dx}(x^3 - 3x^2 + x - 2) = 3x^2 - 6x + 1$

(ii) $\sqrt{1-x} = (1-x)^{\frac{1}{2}}$

$$\frac{d}{dx}(1-x)^{\frac{1}{2}} = \tfrac{1}{2}(1-x)^{-\frac{1}{2}} \times -1$$

$$= -\tfrac{1}{2}(1-x)^{-\frac{1}{2}}$$

$$= -\frac{1}{2\sqrt{1-x}}$$

(iii) $x\sqrt{1-x^2}$

$u = x \quad v = \sqrt{1-x^2} = (1-x^2)^{\frac{1}{2}}$

$u' = 1 \quad v' = \tfrac{1}{2}(1-x^2)^{-\frac{1}{2}} \times -2x = -x(1-x^2)^{-\frac{1}{2}} = -\frac{x}{\sqrt{1-x^2}}$

$$\frac{d}{dx}(x\sqrt{1-x^2}) = vu' + uv'$$

$$= \sqrt{1-x^2} \times 1 + x \times -\frac{x}{\sqrt{1-x^2}}$$

$$= \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}$$

$$= \frac{1-x^2-x^2}{\sqrt{1-x^2}}$$

$$= \frac{1-2x^2}{\sqrt{1-x^2}}$$

(iv) $\frac{x+2}{1-x}$

$u = x + 2 \qquad v = 1 - x$

$u' = 1 \qquad v' = -1$

$$\frac{d}{dx}\left(\frac{x+2}{1-x}\right)=\frac{vu'-uv'}{v^2}$$

$$=\frac{(1-x)\times 1-(x+2)\times -1}{(1-x)^2}$$

$$=\frac{1-x+x+2}{(1-x)^2}=\frac{3}{(1-x)^2}$$

$$=\frac{3}{(x-1)^2}$$

(b) (i) $y=\frac{1}{4}x^2-x$

$4y=x^2-4x$

$4y+4=x^2-4x+4$ (complete the square)

$4(y+1)=(x-2)^2$

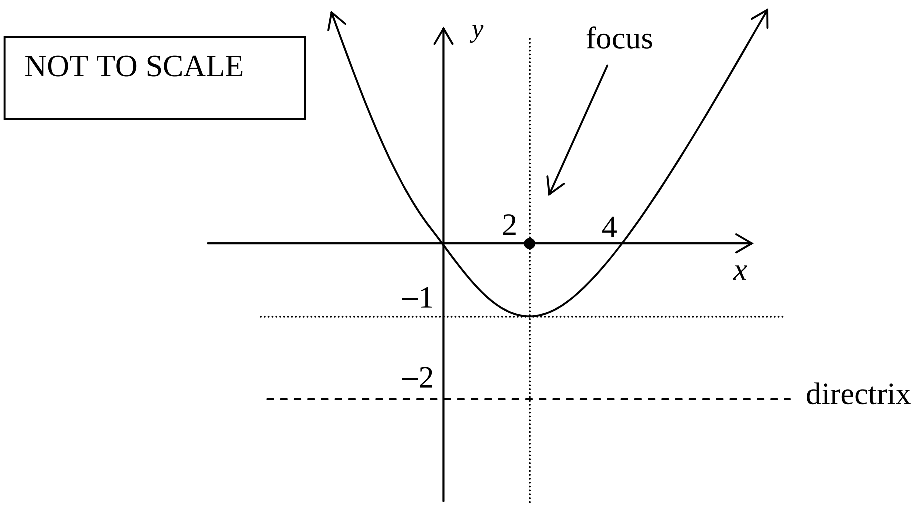
$\therefore (x-2)^2=4(y+1)$

$h=2, a=1, k=-1$

(ii) Vertex $(2,-1)$

Focus (2, 0) (1 unit above the vertex)

(iii)



(c) (i) $\sum_{r=1}^{5}(2r-1)=1+3+5+7+9=25$

(ii) $1+2+4+8+\ldots+256=\sum_{r=1}^{9}2^{r-1}=\sum_{r=0}^{8}2^r$

Question 3

a) $\frac{5}{42}$ of the sample were tagged. We assume a random distribution of fish so if x is the total number of fish then.

$\frac{5}{42} = \frac{75}{x}$ $\therefore$ $x = 75 \times \frac{42}{5} = \underline{630 \text{ fish.}}$

b) i) $x_D =$ is $\frac{5-2}{2}$ ie 1.5

y_D is $\frac{-3+2}{2}$ ie -0.5

So co-ord of D is $\underline{(1\frac{1}{2}, -\frac{1}{2})}$

ii) $\frac{y-4}{x-3} = \frac{2-4}{-2-3}$ $\Rightarrow$ $-5(y-4) = -2(x-3)$

ie $-5y+20 = -2x+6$ OR $\underline{5y-2x-14=0}$

iii) slope of AD is $\frac{4 - -\frac{1}{2}}{3-1\frac{1}{2}} = \frac{9}{2} \times \frac{2}{3} = 3$

So required line is $\frac{y+3}{x-5} = 3$ ie $\underline{y = 3x-18}$

iv) Solve simultaneously. $5y-2x-14=0$ — (1)

$y-3x+18=0$ — (2)

$5\times(2)-(1)$ gives: $0-13x+104=0$

ie $x = 104 = 8$

Sub $x = 8$ into (1) or (2) gives.

$y = 24-18 = 6$ so point of intersection is

$\underline{E(8,6)}$

c) i) $\log 9^n = \log 3 - \log\sqrt{3}$ –(1) find n.

Here the base of the logarithm is arbitrary so lets use base 3. Raising eqn (1) to the power of 3 gives

$9^n = 3 - \sqrt{3}$

ie $(3^2)^n = 3^1 - 3^{\frac{1}{2}}$

ie this is true iff $2n = \frac{1}{2}$

$\therefore n = \frac{1}{2} \div 2 = \frac{1}{4}$

ii) $\log_b a \times \log_a b$ –(1) using the change of base rule.

$\log_b a = \frac{\log_a a}{\log_a b}$ ie sub this into (1) gives.

$\frac{\log_a a}{\log_a b} \times \log_a b = 1$

Question 4

(a) $2x^2+4x-1$

uses $x=-\frac{b}{2a}$

$x=-1$ $y=-3$

(i) $x=-1$

(ii) $y=-3$

(b) $y'=3x^2+6x+3$

$3x^2+6x+3=0$

$x^2+2x+1=0$

$(x+1)(x+1)=0$

$\begin{bmatrix} x=-1 \\ y=-8 \end{bmatrix}$ $y=(-1)^3+3(-1)^2+3(-1)-7$

c) ① $a+11d=2$

② $a+14d=-4$

②-① $3d=-6$

$d=-2$

① $a-22=2$

1st term $a=24$

d) $(x+5)(x-7)=0$

(ii) $x^2-2x-3\geq 0$

$(x-3)(x+1)\geq 0$

$x\leq -1$ or $x\geq 3$

(e)(i) $F=0.4+[0.037+0.00037\ldots]$

(ii) after 1st term geometric series

$a=0.037$ or $\frac{37}{1000}$

$r=\frac{1}{100}$

$S_\infty=\frac{a}{1-r}$

$=\frac{37}{1000}\div\frac{99}{100}$

$=\frac{37}{1000}\times\frac{100}{99}$

$=\frac{37}{990}$

$F=\frac{4}{10}+\frac{37}{990}$

$=\frac{433}{990}$

or

$x=0.4373737\ldots$

$10x=4.373737\ldots$

$1000x=437.3737\ldots$

$990x=433$

$x=\frac{433}{990}$

Question 5

(a) $x^4 - 3x^2 + 2 = 0$

$(x^2 - 1)(x^2 - 2) = 0$

So $x^2 - 1 = 0$ and $x^2 - 2 = 0$

$x^2 = 1$ $\quad$ $x^2 = 2$

$x = \pm 1$ (1) $\quad$ $x = \pm\sqrt{2}$ (1)

(b) $y = x - \frac{1}{x}$

$y = x - x^{-1}$

$\frac{dy}{dx} = 1 + x^{-2} = 1 + \frac{1}{x^2}$

(i) at $x = 2$, $m = 1 + \frac{1}{2^2} = 1 + \frac{1}{4} = 1\frac{1}{4} = \frac{5}{4}$ (1)

(ii) At $x = 2$, $y = 2 - \frac{1}{2} = 1\frac{1}{2} = \frac{3}{2}$ point $(2, \frac{3}{2})$

gradient tangent $\frac{5}{4}$

gradient normal $-\frac{4}{5}$

eq^n tangent $(y - \frac{3}{2}) = \frac{5}{4}(x - 2)$

$4y - 6 = 5x - 10$

$\therefore$ $5x - 4y - 4 = 0$ (1)

eq^n normal $(y - \frac{3}{2}) = -\frac{4}{5}(x - 2)$

$5y - \frac{15}{2} = -4x + 8$

$10y - 15 = -8x + 16$

$\therefore$ $8x + 10y - 31 = 0$ (1)

(c)(i) \$A $r\%$ p.a calculated monthly.
equal instalments \$M.

$$\$B_1 = A + \left(A \times \frac{r\%}{12}\right) - M$$
$$= A + \frac{Ar}{1200} - M$$
$$= A\left(1 + \frac{r}{1200}\right) - M$$
$$\$B_2 = \left[A\left(1+\frac{r}{1200}\right) - M\right] + \left[A\left(1+\frac{r}{1200}\right) - M\right] \times \frac{r}{1200} - M$$
$$= \left[A\left(1+\frac{r}{1200}\right) - M\right]\left[1 + \frac{r}{1200}\right] - M$$
$$= A\left(1+\frac{r}{1200}\right)^2 - \left(1+\frac{r}{1200}\right)M - M$$
$$= A\left(1+\frac{r}{1200}\right)^2 - M\left[1 + \left(1+\frac{r}{1200}\right)\right]$$
$$= A(R)^2 - M(1+R)$$
$$\therefore \$B_n = AR^n - M(1 + R + \cdots + R^{n-1})$$

using $S_n = \frac{r^n - 1}{r-1} = \frac{R \times R^{n-1} - 1}{R-1} = \frac{R^n - 1}{R-1}$

$$\therefore \$B_n = AR^n - M\left(\frac{R^n - 1}{R-1}\right) \text{ as required.}$$

(3)

(c) (ii) \$90,000 $\quad R = 1.00\dot{6} = 1\frac{1}{150}$ $\quad$ 96 instalments

$$0 = 90{,}000 \times \left(1\tfrac{1}{150}\right)^{96} - m\left(\frac{\left(1\frac{1}{150}\right)^{96} - 1}{\left(1\frac{1}{150}\right) - 1}\right)$$

$$m = \frac{90\,000 \times \left(1\frac{1}{150}\right)^{96}}{\left(\frac{\left(1\frac{1}{150}\right)^{96} - 1}{\left(1\frac{1}{150}\right) - 1}\right)} = \$1272.30 \quad (1)$$

(iii) $$\$B_{24} = 90000 \times \left(1\tfrac{1}{150}\right)^{24} - 1272.30\left(\frac{\left(1\frac{1}{150}\right)^{24} - 1}{\left(1\frac{1}{150}\right) - 1}\right)$$

$$= \$72565.1165$$

Now $\$B_{24} = \$72565.1165 - \$10{,}000 = \$62565.1165$ is yet to be paid.

So $$0 = 62565.1165 \times \left(1\tfrac{1}{150}\right)^{n} - 1272.30\left(\frac{\left(1\frac{1}{150}\right)^{n} - 1}{\left(1\frac{1}{150}\right) - 1}\right)$$

$$0 = 62565.1165 \times 1.00\dot{6}^{n} - 190845\left(1.00\dot{6}^{n} - 1\right)$$

$$0 = 62565.1165 \times 1.00\dot{6}^{n} - 190845 \times 1.00\dot{6}^{n} + 190845$$

$$-190845 = -128279.8835 \times 1.00\dot{6}^{n}$$

$$1.487723521 = 1.00\dot{6}^{n}$$

$$\ln 1.487723521 = n \ln 1.00\dot{6}$$

$$n = \frac{\ln 1.487723521}{\ln 1.00\dot{6}}$$

$$n = 59.785$$

$\Rightarrow$ 60 more payments $\quad (3)$